Structure of positive solutions for semilinear elliptic equations with supercritical growth (Shapes and other properties of solutions of PDEs)

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1 Introduction and main results

We study the global bifurcation diagram of the solutions of the supercritical semilinear elliptic Dirichlet problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where $B = \{x \in \mathbf{R}^N : |x| < 1\}$ with $N \geq 3$ and λ is a nonnegative constant. In (1) we assume that f has the form

$$f(u) = u^p + g(u), \tag{2}$$

where $p > p_S := (N+2)/(N-2)$ and $g(u)$ is a lower order term.

By the symmetry result of Gidas-Ni-Nirenberg [9], every regular positive solution u is radially symmetric and $\|u\|_{L^\infty} = u(0)$. It is known that all regular positive solutions can be described as a smooth graph of $\alpha := \|u\|_{L^\infty}$ (see, e.g., [14]). Therefore, the solution set becomes a curve and it is described as $\{(\lambda(\alpha), u_\alpha)\}_{\alpha>0}$ with $\|u_\alpha\|_{L^\infty} = \alpha$. Since $\lambda(\alpha)$ determines the structure of the positive solutions, we mainly study the graph of $\lambda(\alpha)$.

There are several results about bifurcation diagrams of supercritical elliptic equations. Joseph-Lundgren [11] studied the Dirichlet problem

$$\begin{cases} \Delta u + \lambda (u+1)^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3}$$

Define the exponent p_{JL} by

$$p_{JL} := \begin{cases} 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geq 11, \\ \infty, & 2 \leq N \leq 10, \end{cases}$$

which is called the Joseph-Lundgren exponent introduced in [11]. It was shown by [11] that there exists $\lambda^* > 0$ and the following holds: When $p_S < p < p_{JL}$, $\lambda(\alpha)$ oscillates infinitely many times around λ^* and converges to λ^* as $\alpha \to \infty$, and when $p \geq p_{JL}$, $\lambda(\alpha)$ is strictly increasing and converges to λ^* as $\alpha \to \infty$. Note that, by a special change of variables, the problem (3) can be transformed into an autonomous first order system.

The study of the problem

$$\begin{cases} \Delta u + \lambda u + u^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{4}$$

was initiated by Brezis-Nirenberg [1] in the critical case $p = p_S$. Later, the supercritical case $p > p_S$ was studied by Budd-Norbury [3], Budd [4], Merle-Peletier [13], Dolbeault-Flores [8], and Guo-Wei [10]. Note that (4) is transformed into (1) with $f(u) = u + u^p$ by changing $u \mapsto \lambda^{\frac{1}{p-1}} u$. The singular solution of (4) was constructed in [13]. According to [3, 8, 10], the bifurcation curve has infinitely many turning points if $p_S < p < p_{JL}$. In [10] the nonexistence of a turning point for large solutions was proved for a certain range on $p(> p_{JL})$. In general we cannot expect a change of variables that transforms the equation into an autonomous first order system. In [10] they used the intersection number between the regular and singular solution and their Morse indices. In [5, 6, 7] Dancer studied infinitely many turning points for various analytic nonlinear terms, using the analyticity. For other bifurcation diagrams of supercritical problems see [12, 15, 16].

We mainly study the bifurcation curve in the case $p \geq p_{JL}$, using the intersection number. Let us introduce a collection of hypotheses of $f(u)$ in (1).

(f.1) $f \in C^1([0,\infty))$ and $f(u) > 0$ for $u \geq 0$.

(f.2) f has the form (2), where $g(u)$ satisfies

$$|g(u)| \leq C_0 u^{p-\delta} \quad \text{and} \quad |g'(u)| \leq C_0 u^{p-\delta-1} \quad \text{for } u \geq u_0$$

with some constants $u_0 \geq 0$, $\delta > 0$, and $C_0 > 0$.

(f.3) $f(u)$ is convex for $u \geq 0$.

Let $\mathcal{C}$ denote the set of all the regular solution of (1). Assume that (f.1) and (f.2) hold. Then it is known by [14] that $\mathcal{C}$ becomes a curve and is described as

$$\mathcal{C} = \{(\lambda(\alpha), u(r,\alpha)) : \ 0 < \alpha < \infty\} \quad \text{with } u(0,\alpha) = \alpha.$$

Since $f(0) > 0$, $\mathcal{C}$ emanates from $(0,0)$.

By a singular solution u of (1), we mean that $u(r)$ is a classical solution of (1) for $0 < r \le 1$ and satisfies $u(r) \to \infty$ as $r \to 0$. Define $H^1_{0,\mathrm{rad}} = \{u(x) \in H^1_0(B);\ u(x) = u(|x|)\}$. Let $p > p_S$, and assume that (f.1) and (f.2) hold. It was shown by [14] that there exists a singular solution (λ^*, u^*) of (1) such that $u^* \in H^1_{0,\mathrm{rad}}$ and satisfies

$$u^*(r) = A(\sqrt{\lambda^*} r)^{-\theta}(1 + O(r^{\delta\theta})) \quad \text{as } r \downarrow 0, \tag{5}$$

where $\delta > 0$ is the constant in (f.2),

$$\theta = \frac{2}{p-1} \quad \text{and} \quad A := \left\{ \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right\}^{\frac{1}{p-1}}. \tag{6}$$

We show the uniqueness of the singular solution (λ^*, u^*) and the asymptotic behavior of $u(r,\alpha)$ as $\alpha \to \infty$.

Theorem 1. *Let $p > p_S$. Suppose that* (f.1) *and* (f.2) *hold.*

(i) *There exists a unique $\lambda^* > 0$ such that the problem* (1) *with $\lambda = \lambda^*$ has a singular solution u^*. The solution u^* is a unique singular solution of* (1) *with $\lambda = \lambda^*$. Furthermore, $u^* \in H^1_{0,\mathrm{rad}}$ and satisfies* (5) *with* (6).

(ii) *Let $(\lambda(\alpha), u(r,\alpha))$ be a solution of* (1) *with $u(0,\alpha) = \alpha > 0$. Then, as $\alpha \to \infty$,*

$$\lambda(\alpha) \to \lambda^* \quad \text{and} \quad u(r,\alpha) \to u^*(r) \quad \text{in } C^2_{loc}((0,1]), \tag{7}$$

where (λ^*, u^*) is the singular solution in (i).

Remark. The asymptotic properties (7) was shown by Merle-Peletier [13] for the problem (4). We will give a slight simpler proof.

Following the idea by [14], we define three types of bifurcation diagrams according to the intersection number of $\lambda(\alpha)$ and λ^* for $\alpha > 0$. Let $I \subset \mathbf{R}$ be an interval, and let $f \in C(I)$. We define the zero-number of f in I by

$$\begin{aligned} \mathcal{Z}_I(f) \ = \ \sup\{n \in \mathbf{N} :\ & \text{there are } \alpha_1, \ldots, \alpha_{n+1} \in I,\ \alpha_1 < \cdots < \alpha_{n+1} \\ & \text{such that } f(\alpha_i) f(\alpha_{i+1}) < 0 \text{ for } 1 \le i \le n\} \end{aligned}$$

if f changes sign in I, and $\mathcal{Z}_I(f) = 0$ otherwise. By $\mathcal{T}[\mathcal{C}]$ we denote the number of the turning points of $\mathcal{C}$.

Definition. Put $m = \mathcal{Z}_{(0,\infty)}(\lambda(\cdot) - \lambda^*)$.

(i) We say that $\mathcal{C}$ is of Type I if $m = \infty$. As a consequence, if $\mathcal{C}$ is of Type I, then (1) has infinitely many regular solutions for $\lambda = \lambda^*$ and $\mathcal{T}[\mathcal{C}] = \infty$.

(ii) We say that $\mathcal{C}$ is of Type II if $m = 0$.

(iii) We say that $\mathcal{C}$ is of Type III if $1 \leq m < \infty$. As a consequence, if (1) has at least one and finitely many regular solutions for $\lambda = \lambda^*$, then $\mathcal{C}$ is of Type III.

Since $f(0) > 0$, we have $\lambda(\alpha) \to 0$ as $\alpha \to 0$. Then the diagram $\mathcal{C}$ is of type II if $\lambda(\alpha) \leq \lambda^*$ for all $\alpha > 0$. Furthermore, we obtain the following.

Proposition 1. *Assume that* (f.1)-(f.3) *hold. Then $\mathcal{C}$ is of type II if and only if $\lambda(\alpha)$ is strictly increasing and $\lambda(\alpha) \uparrow \lambda^*$ as $\alpha \to \infty$.*

As a consequence, $\mathcal{C}$ is of type II if and only if (1) has a unique regular solution for each $\lambda \in (0, \lambda^*)$ and no regular solution for $\lambda \geq \lambda^*$. In particular, $\mathcal{T}[\mathcal{C}] = 0$. For the problem (3), the diagram $\mathcal{C}$ is of Type I if $p_S < p < p_{JL}$, and Type II if $p \geq p_{JL}$, and Type III does not appear.

Brezis-Vázques [2] studied the problem (1) in a general domain when f is C^1, nondecreasing, convex functions defined on $[0, \infty)$ with

$$f(0) > 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{f(u)}{u} = \infty.$$

It is well known that there exists a finite positive number $\overline{\lambda}$, called the extremal value, such that

(i) for $0 < \lambda < \overline{\lambda}$, there exists a minimal classical solution $u_\lambda \in C^2(\overline{B})$ of (1),

(ii) for $\lambda = \overline{\lambda}$, there exists a weak solution $\overline{u}$ of (1),

(iii) for $\lambda > \overline{\lambda}$, there exists no weak solution of (1).

The solution $\overline{u}$, called the extremal solution, is obtained as the increasing limit of u_λ as $\lambda \uparrow \overline{\lambda}$, and it may be either classical or singular. In the problem (1), if the extremal solution is singular, then $\overline{\lambda} = \lambda^*$, and by (iii), the curve $\mathcal{C}$ is of Type II. Let (λ^*, u^*) be the singular solution of (1). It was shown by [2] that, if $u^* \in H_0^1(B)$, and if u^* is stable in the sense where

$$\int_B \left(|\nabla \phi|^2 - \lambda^* f'(u^*) \phi^2\right) dx \geq 0 \quad \text{for all } \phi \in C_0^1(B),$$

then (λ^*, u^*) is the extremal solution, and hence the curve $\mathcal{C}$ is of Type II.

A partial result about the classification of the bifurcation diagrams was obtained in [14] in terms of Morse index. By $m(u^*)$ we define

$$m(u^*) = \sup\{\dim X : \ X \subset H^1_{0,\mathrm{rad}}(B), \ H[\phi] < 0 \text{ for all } \ \phi \in X \setminus \{0\}\},$$

where

$$H[\phi] = \int_B (|\nabla \phi|^2 - \lambda^* f'(u^*)\phi^2) dx.$$

We call $m(u^*)$ the Morse index of u^*.

Theorem A. [14, Theorems A and B] *Suppose that* $N \geq 3$ *and* (f.1)–(f.2) *hold.*

(i) *If* $p_S < p < p_{JL}$, *then the curve* $\mathcal{C}$ *is of Type I and* $m(u^*) = \infty$.

(ii) *If* $p > p_{JL}$, *then* $0 \leq m(u^*) < \infty$.

In this note, we consider the case $p \geq p_{JL}$ and $N \geq 11$, and investigate the structure of solution curve of (1) by means of the zero number of the solutions to

$$\phi'' + \frac{N-1}{r}\phi' + \lambda^* f'(u^*)\phi = 0 \quad \text{for } 0 < r < 1, \tag{8}$$

where $(\lambda^*, u^*(r))$ be the singular solution of (1). We denote by $z(\phi)$ the the number of the zeros of $\phi(r)$ for $0 < r < 1$. We see that, for any solution ϕ of (8), $z(\phi) = \infty$ if $p_S < p < p_{JL}$, and $0 \leq z(\phi) < \infty$ if $p \geq p_{JL}$. We show the following.

Theorem 2. *Suppose that* (f.1)–(f.3) *hold. Then the following* (i)–(iii) *are equivalent each other.*

(i) *The diagram* C *is of type II.*

(ii) *For any* $\phi \in C^1_{0,\mathrm{rad}}(B)$,

$$\int_B |\nabla \phi|^2 dx \geq \lambda^* \int_B f'(u^*)\phi^2 dx.$$

(iii) *There exists a solution* ϕ *of* (8) *satisfying* $z(\phi) = 0$.

We consider the case where (8) has a solution ϕ satisfying $1 \leq z(\phi) < \infty$. We will see that if $p \geq p_{JL}$, then there exists a unique solution $\phi^*(r) \in C^2(0,1]$ of

$$\begin{cases} (\phi^*)'' + \dfrac{N-1}{r}(\phi^*)' + \lambda^* f'(u^*)\phi^* = 0, & 0 < r < 1, \\ r^{\nu}\phi^*(r) \to 1 \quad \text{as} \quad r \downarrow 0, \end{cases} \tag{9}$$

where

$$\nu = \frac{(1-\varepsilon_p)(N-2)}{2} \quad \text{and} \quad \varepsilon_p = \frac{2}{N-2}\sqrt{\frac{(N-2)^2}{4} - \frac{2p}{p-1}\left(N-2-\frac{2}{p-1}\right)}.$$

Note that $\varepsilon_p \in (0,1)$ if $p > p_{JL}$ and $\varepsilon_p = 0$ if $p = p_{JL}$. By the Strum comparison theorem, we have $|z(\phi_1) - z(\phi_2)| \leq 1$ for any solutions ϕ_1 and ϕ_2 of (8). We see that, for any nontrivial solution ϕ of (8), $z(\phi^*) \leq z(\phi)$. Then, Theorem 2 implies that the curve C is of Type II if and only if $z(\phi^*) = 0$.

We impose the condition on f:

(f.3)' $f(u)$ is convex for $u \geq u_0$ for some $u_0 \geq 0$.

Theorem 3. *Let $N \geq 11$ and $p \geq p_{JL}$. Suppose that* (f.1), (f.2) *and* (f.3)' *hold. Let ϕ^* be the unique solution of the problem* (9). *Assume that $z(\phi^*) \geq 1$. Then $\mathcal{T}[\mathcal{C}] \geq z(\phi^*)$ and* (1) *has at least $z(\phi^*)$ regular solution(s) for $\lambda = \lambda^*$. Assume, in addition, that $\phi^*(1) \neq 0$. Then u is nondegenerate if $\|u\|_{L^\infty}$ is large, and hence, there exist constants $M \geq \tilde{M} > 0$ such that the curve $\{(\lambda(\alpha), u(r,\alpha));\ \alpha > \tilde{M}\}$ has no turning point and $\lambda(\alpha) \neq \lambda^*$ for $\alpha > M$.*

Remark. Note that $z(\phi^*) < \infty$ in the case $N \geq 11$ and $p \geq p_{JL}$.

Corollary 1. *In addition to the hypotheses on N, p and f in Theorem* 3, *assume that f is analytic on $(-\eta, \infty)$ for some $\eta > 0$. Let ϕ^* be the unique solution of the problem* (9). *If $z(\phi^*) \geq 1$ and $\phi^*(1) \neq 0$, then the curve $\mathcal{C}$ is of Type III.*

We see that, if (f.1), (f.2) and (f.3)' hold, then $m(u^*) = z(\phi^*)$. We are led to the following conjecture.

Conjecture. [14, Conjecture 1.4] The bifurcation curve $\mathcal{C}$ has exactly $m(u^*)$ turning point(s) for a certain class of nonlinear terms, i.e., $\mathcal{T}[\mathcal{C}] = m(u^*)$.

Combining Theorem A and Theorems 2 and 3, we can classify bifurcation diagrams as Table 1 shows.

Table 1 tells us that the structure of the regular solutions of (1) is encoded in the singular solution. From the viewpoint of the Morse index of the singular solution, a Type III bifurcation diagram is an intermediate case between Type I and Type II bifurcation diagrams.

$$
\left\{
\begin{array}{lcccc}
p_S < p < p_{JL} & \underset{\text{Theorem A (i)}}{\Longrightarrow} & m(u^*) = \infty & \text{and} & \begin{array}{c}\text{Type I}\\ (\mathcal{T}[\mathcal{C}]=\infty)\end{array} \\
p \ge p_{JL} & \underset{\text{Theorem A (ii)}}{\Longrightarrow} & \left\{\begin{array}{c} m(u^*)=0 \\ \text{or} \\ 1\le m(u^*)<\infty\end{array}\right. & \begin{array}{c}\underset{\text{Theorem 2}}{\Longrightarrow} \\ \\ \underset{\text{Corollary 1}}{\Longrightarrow}\end{array} & \begin{array}{c}\text{Type II}\\(\mathcal{T}[\mathcal{C}]=0)\\ \text{Type III}\\(m(u^*)\le \mathcal{T}[\mathcal{C}]<\infty)\end{array}
\end{array}\right.
$$

Table 1: Classification of bifurcation diagrams for supercritical elliptic equations with power growth.

2 Sufficient conditions for Types II and III

We will show some sufficient conditions for Types II and III.

Theorem 4. *Let* $N \ge 11$ *and* $p \ge p_{JL}$. *Suppose that* f *satisfies* (f.1)–(f.3). *Assume in* (2) *that* $g(u) \ge 0$ *for* $u \ge 0$ *and*

$$g'(u) \le C_A u^{p-1} \quad \textit{for } u \ge 0, \tag{10}$$

where

$$C_A = \frac{\frac{(N-2)^2}{4} - pA^{p-1}}{A^{p-1}}.$$

Then the curve $\mathcal{C}$ *is of Type II.*

Remark. (i) We see that $C_A > 0$ if $p > p_{JL}$ and $C_A = 0$ if $p = p_{JL}$. Thus, in the case $p = p_{JL}$, the condition (10) leads that $g'(u) \le 0$ for $u \ge 0$.

(ii) Let $p > p_{JL}$. Since we assume (f.2), the inequality (10) is satisfied for sufficiently large u automatically. Thus the condition (10) require that inequality holds for $u \in [0, u_0]$ with some $u_0 > 0$.

For $a \ge 0$, define $f_a(u) = f(u + a)$ for $u \ge 0$. Let us consider the problem

$$\begin{cases} \Delta u + \lambda f_a(u) = 0, & x \in B, \\ u > 0, & x \in B, \\ u = 0, & x \in \partial B. \end{cases} \tag{11}$$

We obtain the following.

Theorem 5. *Let* $N \ge 11$ *and* $p \ge p_{JL}$. *Assume that* f *satisfies* (f.1), (f.2) *and* (f.3)'. *Then there exists* $a_0 \ge 0$ *such that, for all* $a \ge a_0$, *the curve* $\mathcal{C}$ *of the problem* (11) *is of Type II.*

To show examples of Type III bifurcation diagram, we impose the condition on f:

(f.1)' $f \in C^1[0,\infty)$, $f(u) > 0$ for $u > 0$, and $f(0) = 0$.

Define $F(u)$ by

$$F(u) = \int_0^u f(t)dt \quad \text{for } u \geq 0.$$

We obtain the following.

Theorem 6. *Suppose that $p \geq p_{JL}$ and* (f.1)', (f.2), (f.3) *hold and $f(u)$ is analytic for $u > 0$. Assume that $g(u)$ in* (2) *satisfies*

$$g(u) = u^q + O(u^{q+\delta_0}) \quad \textit{as } u \to 0$$

and

$$g'(u) = qu^{q-1} + O(u^{q-1+\delta_0}) \quad \textit{as } u \to 0$$

with some constants $q \in (p_S, p_{JL})$ and $\delta_0 > 0$. Assume, in addition, that

$$(q+1)F(u) \leq uf(u) \quad \textit{for } u \geq 0.$$

Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_1 > a_2 > \cdots > a_n > \cdots > 0$ and the following holds: If $a_{n+1} < a < a_n$ for some $n \geq 1$, then the problem (11) *has a Type III bifurcation diagram and $n \leq \mathcal{T}[\mathcal{C}] < \infty$ hold.*

A typical example of f in Theorem 6 is given by

$$f(u) = u^p + u^q \quad \text{with } p_S < q < p_{JL} \leq p. \tag{12}$$

By changing the variables $u \mapsto au$ and $\lambda \mapsto a^{1-p}\lambda$, we see that (11) with (12) is equivalent to the problem

$$\begin{cases} \Delta u + \lambda\{(u+1)^p + b(u+1)^q\} = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{13}$$

where $b := a^{q-p}$. We obtain the following:

Corollary 2. *Let $\{a_n\}_{n=1}^{\infty}$ be as in Theorem 6. If $a_n^{q-p} < b < a_{n+1}^{q-p}$ for some $n \geq 1$, then the problem* (13) *has Type III bifurcation diagram and $n \leq \mathcal{T}[\mathcal{C}] < \infty$.*

We can intuitively understand Corollary 2 in the following way: If $b > 0$ is small, then (13) is close to (3) with $p \geq p_{JL}$, and hence, $\mathcal{C}$ is of Type II. When b is large, the term $b(u+1)^q$ with $p_S < q < p_{JL}$ is dominant for a relatively small solution u and $\mathcal{C}$ has turning point(s). However, if u is large, then $(u+1)^p$ with $p \geq p_{JL}$ becomes dominant and u is nondegenerate. Hence, this is an intermediate case. Moreover, the lower bound of $\mathcal{T}[\mathcal{C}]$ can be controlled by $b = a^{q-p}$.

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